Advanced Mathematical Models & Applications
Vol.7, No.2, 2022, pp.130-145

ON SOME PROPERTIES AND APPLICATIONS OF THE GENERALIZED m-PARAMETER MITTAG-LEFFLER FUNCTION

Ankita Chandola[1], Rupakshi Mishra Pandey[1], Ritu Agarwal[2], Laxmi Rathour[3], Vishnu Narayan Mishra[4*]

[1]Amity Institute of Applied Sciences, Amity University, Uttar Pradesh, India
[2]Malaviya National Institute of Technology, Jaipur, India
[3]Ward Number 16, Bhagatbandh, Annupur 484 224, Madhya Pradesh
[4]Department of Mathematics, Indira Gandhi National Tribal University, Lalpur, Amarkantak, Anuppur, Madhya Pradesh, India

Abstract. The Mittag-Leffler function plays a very important role in the theory of fractional differential equation, fractional kinetics, medical science and various applications in physics and engineering. In this article, we use the m-parameter Mittag-Leffler function, explore its particular cases and visually illustrate the contour plots graphically exhibiting the behaviour of the m-parameter Mittag-Leffler function for different values of the parameters using MATHEMATICA-12. This concept of contour plotting and analyticity can be utilised to construct a numerical evaluation technique of the m-parameter Mittag-Leffler function. The same approach may be followed for other functions of the hypergeometric type and with some small modifications can be applied for the Wright function which plays a very important role in the theory of partial differential equations of fractional order. Further we have developed a new generalized form of fractional kinetic equation and obtained its solution using Natural transform as an application of m-parameter Mittag-Leffler function.

Keywords: m-parameter Mittag-Leffler function, Contour plots, Analyticity, Generalized fractional Kinetic equation, Natural transform, Fractional operator.

AMS Subject Classification: 33E12, 35A20, 26A33, 44A20.

Corresponding author: Vishnu Narayan Mishra, Department of Mathematics, Indira Gandhi National Tribal University, Lalpur, Amarkantak, Anuppur, Madhya Pradesh, India, e-mail: *vnm@igntu.ac.in, vishnunarayanmishra@gmail.com*

Received: 26 April 2022; Revised: 5 July 2022; Accepted: 19 July 2022; Published: 5 August 2022.

1 Introduction

The Mittag-Leffler function play a major role in physics and engineering problems. It is also used in fractional calculus and its applications (Kumar et al., 2020). The Mittag-Leffler function arises in the solution of fractional order integral equations or fractional order differential equations. It is mostly used in the area of applied sciences like rheology, fluid flow and electric network, fractal kinetics (Atangana, 2017), medical science (Ghanbari et al., 2020), fractional and fractal calculus and its applications (Nisar et al., 2020). It arises in the investigation of random walks, Levy flights, super diffusive transport, fractional generalization of kinetic equation and in the study of complex systems.

The kinetic equations describe the continuity of motion of substance. Due to the effectiveness and a great importance of the kinetic equation in certain astrophysical problems we develop a further generalized form of the fractional kinetic equation involving the m-parameter Mittag-Leffler function.

Haubold and Mathai (2000) discussed the fractional differential equation between the rate of change of the reaction, the destruction rate and the production rate. Let $\mathfrak{M} = (\mathfrak{M}_\tau)$ be the arbitrary reaction which is defined by a time dependent quantity. Then the fractional differential equation is given by

$$\frac{d\mathfrak{M}}{d\tau} = -d(\mathfrak{M}_\tau) + r(\mathfrak{M}_\tau), \tag{1}$$

where
$d = d(\mathfrak{M})$ is the rate of destruction,
$r = r(\mathfrak{M})$ is the rate of production and
$(\mathfrak{M}_\tau)$ is the function defined by $\mathfrak{M}_\tau(\tau^*) = \mathfrak{M}(\tau - \tau^*)$, $\tau^* > 0$.

As a particular case of the above equation (1), we have

$$\frac{d\mathfrak{M}_j}{d\tau} = -p_j \mathfrak{M}_j(\tau), \tag{2}$$

with $\mathfrak{M}_j(\tau = 0) = \mathfrak{M}_0$ is the number density of the species j at time $\tau = 0$ and $p_j > 0$. The solution of equation (2) is given by

$$\mathfrak{M}_j(\tau) = \mathfrak{M}_0 e^{-p_j \tau}. \tag{3}$$

On integrating equation (3), we get

$$\mathfrak{M}(\tau) - \mathfrak{M}_0 = -p_0 \mathfrak{D}_\tau^{-1} \mathfrak{M}(\tau), \tag{4}$$

where ${}_0\mathfrak{D}_\tau^{-1}$ is the particular case of Riemann-Liouville integral operator ${}_0\mathfrak{D}_\tau^{-\xi}$ given by

$$({}_0\mathfrak{D}_\tau^{-\xi} g)(\tau) = \frac{1}{\Gamma(\xi)} \int_0^\tau (\tau - u)^{\xi - 1} g(u) d(u), \qquad \Re(\xi) > 0. \tag{5}$$

where Γ is gamma function (Rainville, 1971, Eq. (8), p.9)

$$\Gamma(m) = \int_0^\infty t^{m-1} e^{-t} dt, \qquad \Re(m) > 0.$$

For more details on Riemann-Liouville fractional operator, readers can refer Kilbas et al. (2006); Liouville (1823).

Also, Haubold and Mathai (2000) gave the fractional generalization of the standard kinetic equation (4)

$$\mathfrak{M}(\tau) - \mathfrak{M}_0 = -p^\xi {}_0\mathfrak{D}_\tau^{-1} \mathfrak{M}(\tau) \tag{6}$$

whose solution is given by

$$\mathfrak{M}(\tau) = \mathfrak{M}_0 \sum_{s=0}^{\infty} \frac{(-1)^s}{\Gamma(\xi s + 1)} (p\tau)^{\xi s}. \tag{7}$$

Saxena and Kalla (2008) gave the following fractional equation

$$\mathfrak{M}(\tau) - \mathfrak{M}_0 g(\tau) = -p^\xi {}_0\mathfrak{D}_\tau^{-\xi} \mathfrak{M}(\tau), \qquad \Re(\xi) > 0. \tag{8}$$

We have used Natural transform in our work since it is a generalized version for the Laplace transform (Spiegel, 1965) and the Sumudu transforms (Watugala, 1993). Khan and Khan (2008) first introduced Natural transform.

The Natural transform of the function $g(\tau)$ is given as (Khan and Khan, 2008, Eq. (1), p.127)

$$G(s, u) = N[g(\tau); s, u] = \int_0^\infty e^{-s\tau} g(u\tau) d\tau, \tag{9}$$

where $\Re(s) > 0$, s is a complex number, $u \in (-t_1, t_2)$. The real function $g(\tau) > 0$ and $g(\tau) = 0$ for $\tau < 0$ is sectionwise continuous, exponential order and defined in the set

$$A = \{g(\tau) : \exists H, t_1, t_2 > 0, |g(\tau)| < He^{\frac{|\tau|}{t_j}}, \tau \in (-1)^j \times [0, \infty)\}.$$

Under particular values of u and s, the Natural transform reduces to Laplace and Sumudu transforms:

(i) When $u = 1$, the Natural transform reduces to Laplace transform Spiegel (1965) which is given by

$$G(s) = L[g(\tau); s] = \int_0^\infty e^{-s\tau} g(\tau) d\tau,$$

where s is a complex number, $\Re(s) > 0$, $g(\tau)$ is of exponential order and piecewise continuous.

(ii) When $s = 1$, the Natural transform reduces to Sumudu transform Watugala (1993) which is given by

$$G(u) = S[g(\tau); u] = \int_0^\infty e^{-\tau} g(u\tau) d\tau, \qquad u \in (-t_1, t_2),$$

where $g(\tau)$ is of exponential order and piecewise continuous.

For studying various types of generalizations of functions, their properties, applications and generalizations of fractional kinetic equations with their solutions readers may refer to the following papers Agarwal et al. (2018); Chandola et al. (2020, 2021a,b); Nisar (2020).

The generalized Mittag-Leffler function with m-parameters (Agarwal et al., 2021) is defined as

$$E^{\mu_1,\nu_1;\mu_2,\nu_2;\ldots;\mu_r,\nu_r}_{\alpha,\rho;\beta_1,\kappa_1;\beta_2,\kappa_2;\ldots;\beta_p,\kappa_p}(t) = E^{(\boldsymbol{\mu},\boldsymbol{\nu})_r}_{\alpha,\rho;(\boldsymbol{\beta},\boldsymbol{\kappa})_p}(t) = \sum_{n=0}^{\infty} \frac{(\mu_1)_{\nu_1 n}(\mu_2)_{\nu_2 n}\ldots(\mu_r)_{\nu_r n}}{\Gamma(\alpha n+\rho)(\beta_1)_{\kappa_1 n}\ldots(\beta_p)_{\kappa_p n}} t^n, \tag{10}$$

where $(\boldsymbol{\mu},\boldsymbol{\nu})_r = [\mu_1, \nu_1; \mu_2, \nu_2; \ldots; \mu_r, \nu_r]$, $(\boldsymbol{\beta},\boldsymbol{\kappa})_p = [\beta_1, \kappa_1; \beta_2, \kappa_2; \ldots; \beta_p, \kappa_p]$, $r+p = m-2$, m is any positive integer, t is a complex variable, $\mu_i, \nu_i, \alpha, \rho, \beta_j, \kappa_j \in \mathbb{C}$, with $\min \Re\{\alpha, \rho, \mu_i, \nu_i, \beta_j, \kappa_j\} > 0$ for $i = 1, ..., r$; $j = 1, ..., p$.

The generalized Mittag-Leffler function with m parameters reduces to the following *special cases* on giving specific values to the various parameters:

(a) For $r = p = 0$ and $\rho = 1$ equation (10) reduces to the Gosta Mittag-Leffler function Mittag-Leffler (1903) given by

$$E_\alpha(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\alpha n + 1)}, \tag{11}$$

where $\alpha \geq 0$. E_α is entire function of the complex variable t, and Γ is gamma function (Rainville, 1971, Eq. (8), p.9).

(b) For $r = p = 0$ equation (10) reduces to the Wiman function (Wiman, 1905, p.191) given by

$$E_{\alpha,\rho}(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\alpha n + \rho)}, \quad \alpha, \rho \in \mathbb{C}, \quad \Re(\alpha), \Re(\rho) > 0. \tag{12}$$

(c) For $r = p = 1$ and $\nu_1 = 1, \beta_1 = 1, \kappa_1 = 1$ equation (10) reduces to the Prabhakar function (Prabhakar, 1971, Eq. (1.3), p.7) given by

$$E^{\mu}_{\alpha,\rho}(t) = \sum_{n=0}^{\infty} \frac{(\mu)_n t^n}{\Gamma(\alpha n + \rho) n!}, \quad \Re(\alpha) > 0, \Re(\rho) > 0, \ \mu > 0 \tag{13}$$

where $(\mu)_n$ is the Pochhammer symbol defined by (Rainville, 1971, Eq.(1), p.22; Eq.(3), p.23)

$$(\mu)_n = \frac{\Gamma(\mu+n)}{\Gamma(\mu)} = \begin{cases} \prod_{r=1}^{n}(\mu+r-1), & n \in \mathbb{N} \\ 1, & n=0 \end{cases} \qquad \mu \neq 0, -1, -2, \ldots.$$

(d) For $r = p = 1$ and $\beta_1 = 1, \kappa_1 = 1$ equation (10) reduces to the generalization of Prabhakar function given by Shukla and Prajapati (Shukla and Prajapati, 2007, Eq. (1.4), p.798) and defined as

$$E_{\alpha,\rho}^{\mu,\nu}(t) = \sum_{n=0}^{\infty} \frac{(\mu)_{\nu n} t^n}{\Gamma(\alpha n+\rho) n!}, \tag{14}$$

where $min\{\Re(\alpha), \Re(\rho), \Re(\mu)\} > 0$, $\alpha, \rho, \mu, \nu \in \mathbb{C}$, $(\mu)_{\nu n} = \frac{\Gamma(\mu+\nu n)}{\Gamma(\mu)}$ is the generalized Pochhammer symbol, which in particular reduces to $(\mu)_{\nu n} = \nu^{\nu n} \prod_{i=1}^{\nu} \left(\frac{\mu+i-1}{\nu}\right)_n$ if $\nu \in \mathbb{N}$.

(e) For $r = p = 1$ and $\kappa_1 = 1$ equation (10) reduces to the generalization given by Khan and Ahmed (Khan and Ahmed, 2013, Eq. (1.7), p.2) as

$$E_{\alpha,\rho;\beta}^{\mu,\nu}(t) = \sum_{n=0}^{\infty} \frac{(\mu)_{\nu n} t^n}{\Gamma(\alpha n+\rho)(\beta)_n}, \tag{15}$$

where $\min\{\Re(\alpha), \Re(\rho), \Re(\beta), \Re(\mu)\} > 0$ and $\nu \in (0,1) \cup \mathbb{N}$.

(f) For $r = p = 2$ equation (10) reduces to the further generalization given by Khan and Ahmed (Khan and Ahmed, 2013, Eq. (1.9), p.2), defined by

$$E_{\alpha,\rho;\beta_1,\kappa_1;\beta_2,\kappa_2}^{\mu_1,\nu_1;\mu_2,\nu_2}(t) = \sum_{n=0}^{\infty} \frac{(\mu_1)_{\nu_1 n}(\mu_2)_{\nu_2 n} t^n}{\Gamma(\alpha n+\rho)(\beta_1)_{\kappa_1 n}(\beta_2)_{\kappa_2 n}}, \tag{16}$$

where $\alpha, \rho, \mu_1, \nu_1, \mu_2, \beta_1, \kappa_1, \beta_2 \in \mathbb{C}$, $\min\{\Re(\alpha), \Re(\rho), \Re(\mu_1), \Re(\nu_1), \Re(\mu_2), \Re(\beta_1), \Re(\kappa_1), \Re(\beta_2)\}$ > 0, $\nu_2, \kappa_2 > 0$, $\Re(\alpha) + \kappa_2 \geq \nu_2$.

In this paper, we present contour plots of particular cases of m-parameter Mittag-Leffler function which will be useful in future to define a numerical algorithm to discuss the behaviour of m-parameter Mittag-Leffler function. Also, as an application of m-parameter Mittag-Leffler function we will generalize the standard kinetic equation to a fractional kinetic equation using m-parameter Mittag-Leffler function and use Natural transform to find the solution of this fractional kinetic equation.

2 Contour Lines

In this section, we take some particular cases of m-parameter Mittag-Leffler function and represent its contour plots graphically. Also, we will discuss the analyticity of those functions.

Let us denote the contour plot of real and imaginary part of $E_{\alpha,\rho;(\boldsymbol{\beta},\boldsymbol{\kappa})_p}^{(\boldsymbol{\mu},\boldsymbol{\nu})_r}(t)$.

1. Contour Plot for real part of $E_{\alpha,\rho;(\boldsymbol{\beta},\boldsymbol{\kappa})_p}^{(\boldsymbol{\mu},\boldsymbol{\nu})_r}(t)$ is

$$^{\Re}\mathcal{C}_{\alpha,\rho;(\boldsymbol{\beta},\boldsymbol{\kappa})_p}^{(\boldsymbol{\mu},\boldsymbol{\nu})_r}(v) = \{t \in \mathbb{C} : \Re[E_{\alpha,\rho;(\boldsymbol{\beta},\boldsymbol{\kappa})_p}^{(\boldsymbol{\mu},\boldsymbol{\nu})_r}(t)] = v\}. \tag{17}$$

2. Contour Plot for imaginary part of $E^{(\boldsymbol{\mu},\boldsymbol{\nu})_r}_{\alpha,\rho;(\boldsymbol{\beta},\boldsymbol{\kappa})_p}(t)$ is

$$^{\Im}\mathcal{C}^{(\boldsymbol{\mu},\boldsymbol{\nu})_r}_{\alpha,\rho;(\boldsymbol{\beta},\boldsymbol{\kappa})_p}(v) = \{t \in \mathbb{C} : \Im[E^{(\boldsymbol{\mu},\boldsymbol{\nu})_r}_{\alpha,\rho;(\boldsymbol{\beta},\boldsymbol{\kappa})_p}(t)] = v\}. \tag{18}$$

Fixing $r = p = 4$ (i.e., $m = 10$) in the m-parameter Mittag-Leffler function (10) we have

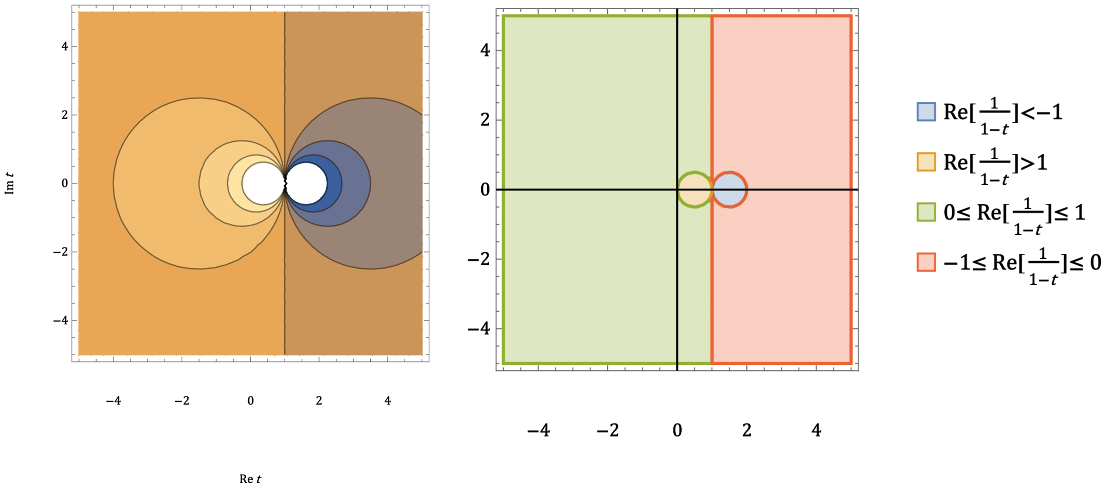
$$E^{\mu_1,\nu_1,\mu_2,\nu_2,\mu_3,\nu_3,\mu_4,\nu_4}_{\alpha,\rho,\beta_1,\kappa_1,\beta_2,\kappa_2,\beta_3,\kappa_3,\beta_4,\kappa_4}(t) = \sum_{n=0}^{\infty} \frac{(\mu_1)_{\nu_1 n}(\mu_2)_{\nu_2 n}(\mu_3)_{\nu_3 n}(\mu_4)_{\nu_4 n}}{\Gamma(\alpha n+\rho)(\beta_1)_{\kappa_1 n}(\beta_2)_{\kappa_2 n}(\beta_3)_{\kappa_3 n}(\beta_4)_{\kappa_4 n}} t^n. \tag{19}$$

We now give particular values to the parameters and discuss the contour plot and the analyticity of the function.

Table 1: Contour plots and Analyticity of m-parameter Mittag-Leffler function.

Particular values			
Parameters	Case 1	Case 2	Case 3
α	0	1	1
ρ	1	1	2
β_1	1	1	1
κ_1	1	1	1
β_2	1	1	1
κ_2	1	1	1
β_3	1	1	1
κ_3	1	1	1
β_4	1	1	1
κ_4	1	1	1
μ_1	1	1	1
ν_1	1	1	1
μ_2	1	1	1
ν_2	1	1	1
μ_3	1	1	1
ν_3	1	1	1
μ_4	1	1	1
ν_4	1	1	1
Function	$\frac{1}{1-t}$	e^t	$\frac{-1+e^t}{t}$
Analyticity	Simple pole at $t = 1$	Entire	Pole at $t = 0$, zeros at $t = 2k\pi\iota, k \in \mathbb{Z}$

Case 1.



(a) Contour Plot. **(b)** Complex Region Plot.

Figure 1: $\Re[E^{1,1,1,1,1,1,1,1,1}_{0,1,1,1,1,1,1,1,1,1,1}(t)] = \Re\left(\frac{1}{1-t}\right)$.

Observation

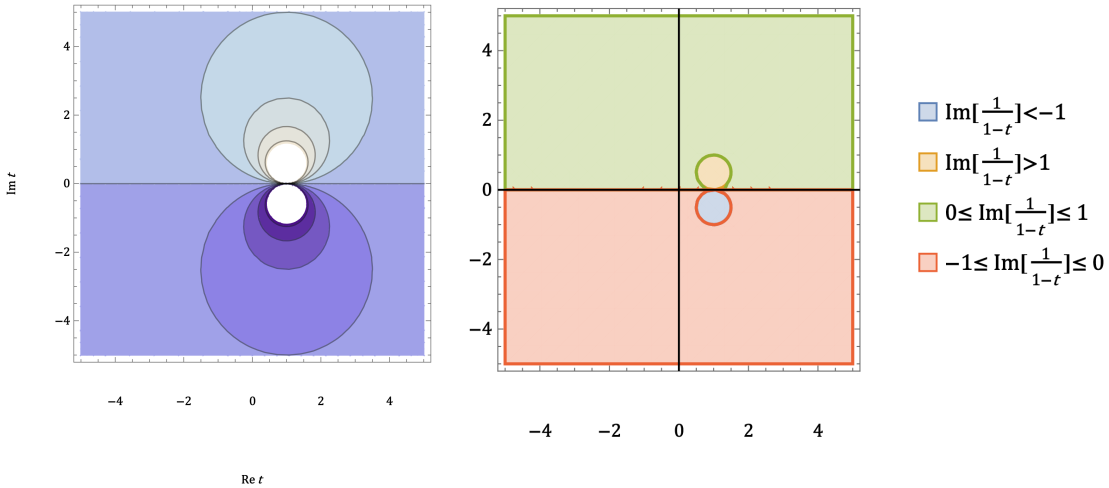
1. The contour line ${}^{\Re}\mathcal{C}^{1,1,1,1,1,1,1,1,1}_{0,1,1,1,1,1,1,1,1,1,1}(0)$ is the vertical line $\Re(t) = 1$, separating the left and the right half planes.

2. The white disc on the left of the line $\Re(t) = 1$ represents $\{t \in \mathbb{C} : \Re[E^{1,1,1,1,1,1,1,1,1}_{0,1,1,1,1,1,1,1,1,1,1}(t)] > 1\}$.

3. The white disc on the right of the line $\Re(t) = 1$ represents $\{t \in \mathbb{C} : \Re[E^{1,1,1,1,1,1,1,1,1}_{0,1,1,1,1,1,1,1,1,1,1}(t)] < -1\}$.

4. The contour lines ${}^{\Re}\mathcal{C}^{1,1,1,1,1,1,1,1,1}_{0,1,1,1,1,1,1,1,1,1,1}(1)$ is the boundary of the circle of the white disc on the left.

5. The contour lines ${}^{\Re}\mathcal{C}^{1,1,1,1,1,1,1,1,1}_{0,1,1,1,1,1,1,1,1,1,1}(-1)$ is the boundary of the circle of the white disc on the right.

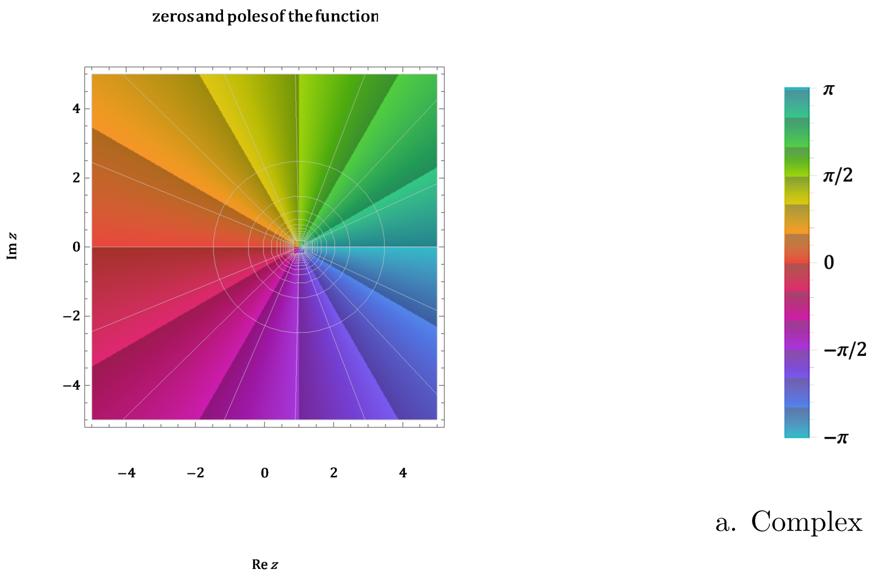


(a) Contour Plot. **(b)** Complex Region Plot.

Figure 2: $\Im[E^{1,1,1,1,1,1,1,1,1}_{0,1,1,1,1,1,1,1,1,1,1}(t)] = \Im\left(\frac{1}{1-t}\right)$.

Observation

1. The contour line ${}^{\Im}\mathcal{C}^{1,1,1,1,1,1,1,1,1}_{0,1,1,1,1,1,1,1,1,1,1}(0)$ is the horizontal line $\Im(t) = 0$, separating the upper and the lower half planes.
2. The white disc above the line $\Im(t) = 0$ represents $\{t \in \mathbb{C} : \Im[E^{1,1,1,1,1,1,1,1,1}_{0,1,1,1,1,1,1,1,1,1,1}(t)] > 1\}$.
3. The white disc below the line $\Im(t) = 0$ represents $\{t \in \mathbb{C} : \Im[E^{1,1,1,1,1,1,1,1,1}_{0,1,1,1,1,1,1,1,1,1,1}(t)] < -1\}$.
4. The contour lines ${}^{\Im}\mathcal{C}^{1,1,1,1,1,1,1,1,1}_{0,1,1,1,1,1,1,1,1,1,1}(1)$ is the boundary of the circle of the white disc on the upper half plane.
5. The contour lines ${}^{\Im}\mathcal{C}^{1,1,1,1,1,1,1,1,1}_{0,1,1,1,1,1,1,1,1,1,1}(-1)$ is the boundary of the circle of the white disc on the lower half plane.



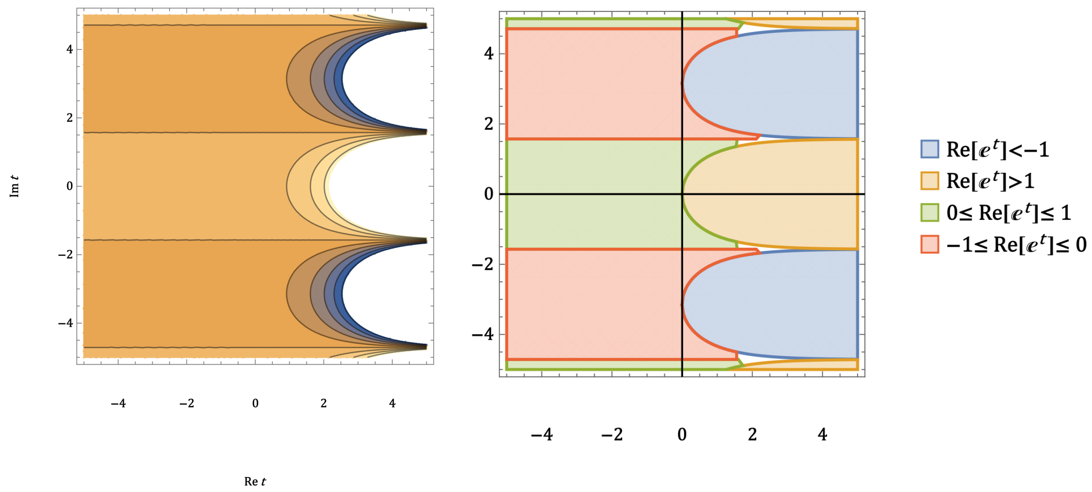
a. Complex Plot.

Figure 3: Zeros and Poles of $[E^{1,1,1,1,1,1,1,1,1}_{0,1,1,1,1,1,1,1,1,1,1}(t)] = \frac{1}{1-t}$.

Observation

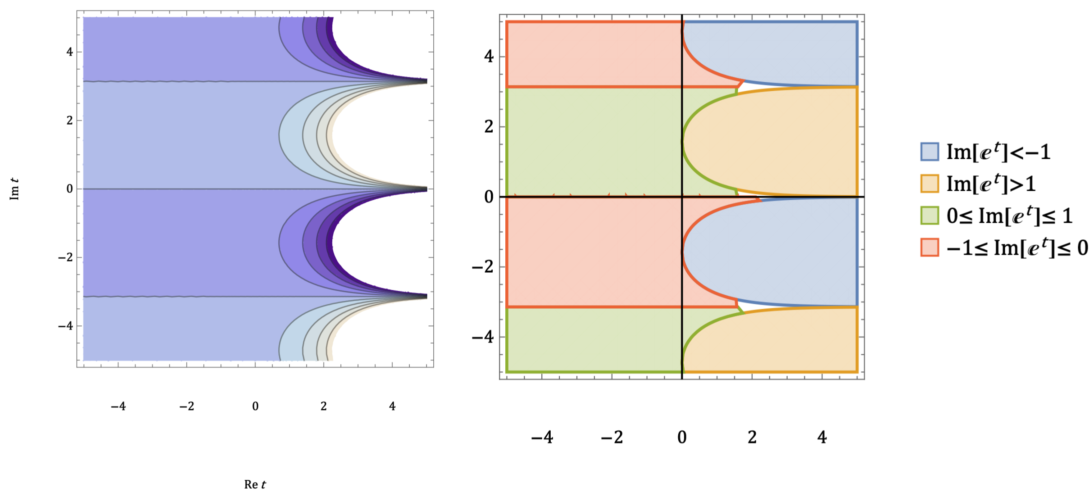
The function $E^{1,1,1,1,1,1,1,1}_{0,1,1,1,1,1,1,1,1,1}(t)$ is not entire, but it can be analytically continued to all of $\mathbb{C} \setminus \{1\}$ and has a simple pole at $t = 1$.

Case 2.



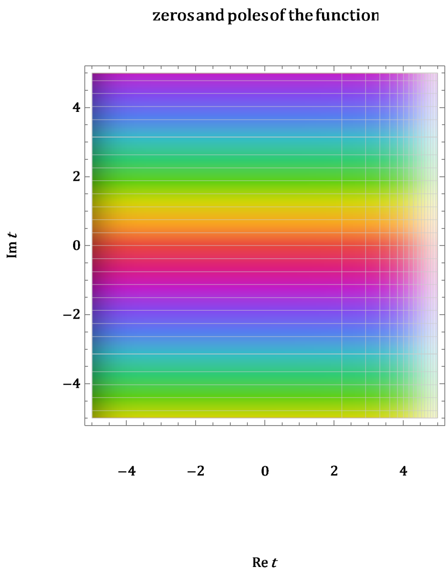
(a) Contour Plot. **(b)** Complex Region Plot.

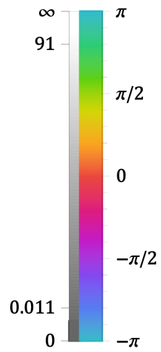
Figure 4: $\Re[E^{1,1,1,1,1,1,1,1}_{1,1,1,1,1,1,1,1,1,1}(t)] = \Re\,(e^t)$.



(a) Contour Plot. **(b)** Complex Region Plot.

Figure 5: $\Im[E^{1,1,1,1,1,1,1,1}_{1,1,1,1,1,1,1,1,1,1}(t)] = \Im\,(e^t)$.





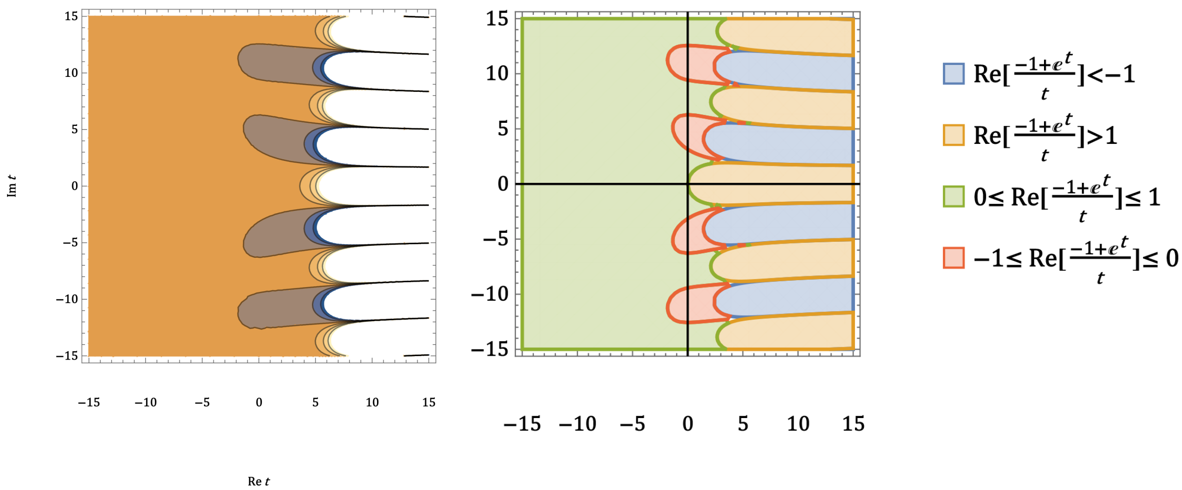
a. Complex Plot.

Figure 6: Zeros and Poles of $[E^{1,1,1,1,1,1,1,1,1}_{1,1,1,1,1,1,1,1,1,1,1}(t)] = e^t$.

Observation

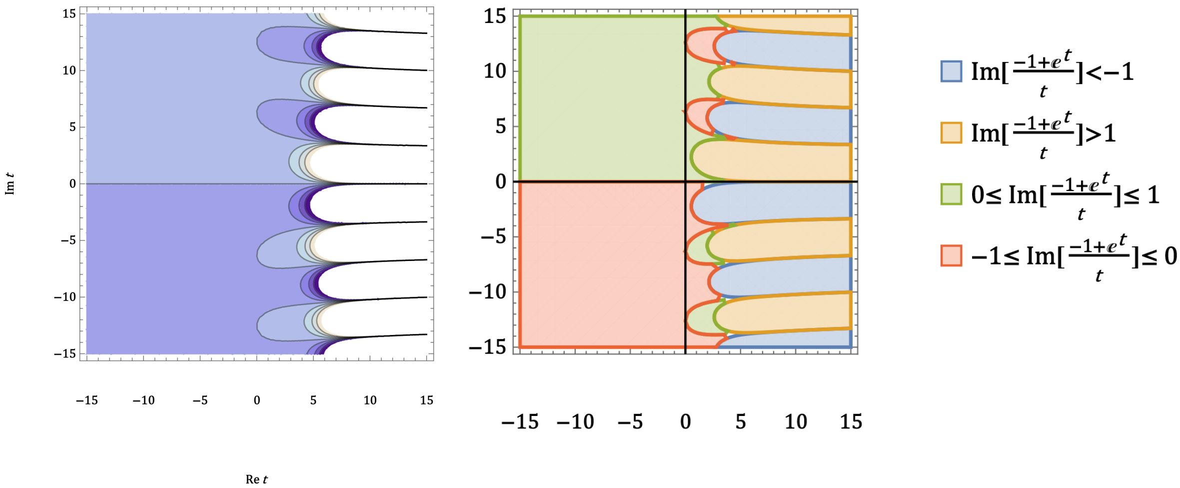
For case 2, that is when α changes from 0 to 1, there is a drastic change in the contour plot of the function. The function is entire and wedge is the right half plane with lobes running parallel to the real axis.

Case 3.



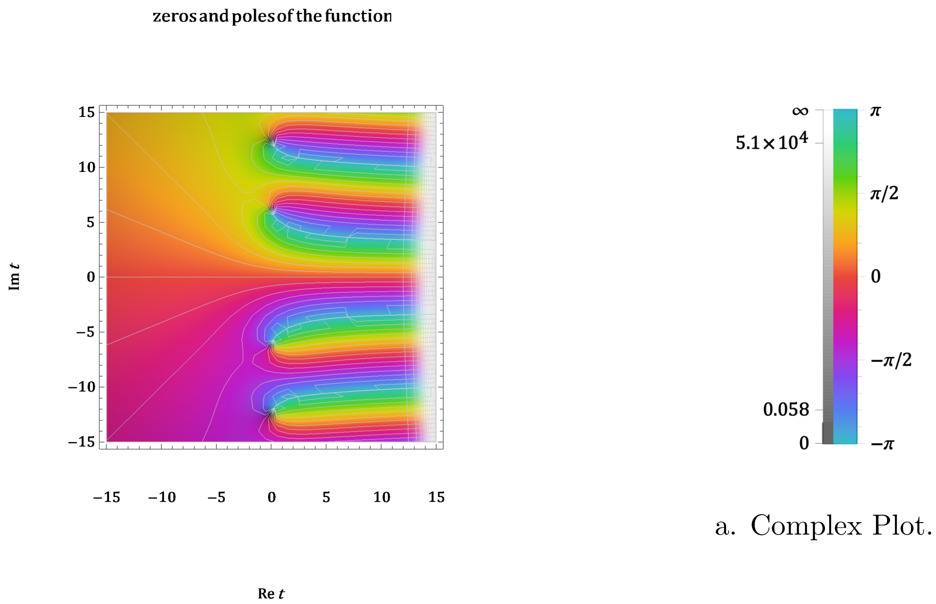
(a) Contour Plot **(b)** Complex Region Plot

Figure 7: $\Re[E^{1,1,1,1,1,1,1,1,1}_{1,2,1,1,1,1,1,1,1,1,1}(t)] = \Re\left(\frac{-1+e^t}{t}\right)$.



(a) Contour Plot. **(b)** Complex Region Plot.

Figure 8: $\Im[E^{1,1,1,1,1,1,1,1}_{1,2,1,1,1,1,1,1,1,1}(t)] = \Im\left(\frac{-1+e^t}{t}\right)$.



a. Complex Plot.

Figure 9: Zeros and Poles of $[E^{1,1,1,1,1,1,1,1,1}_{1,2,1,1,1,1,1,1,1,1}(t)] = \frac{-1+e^t}{t}$.

Observation

For case 3, keeping $\alpha = 1$, we change ρ from 1 to 2. We observe that the fingers extend more and more towards the left half plane. Also, the function $E^{1,1,1,1,1,1,1,1,1}_{1,2,1,1,1,1,1,1,1,1}(t)$ has poles at $t = 0$ and zeros at $t = 2k\pi\iota, k \in \mathbb{Z}$.

Hence, the particular cases considered shows that the function changes its analyticity based on the values of parameters. Similarly by taking different values of m and particular values of the parameters, we can study the contour plots and discuss about the analyticity of various functions.

The complex and the contour plots of m-parameter Mittag-Leffler function will be useful in developing a numerical algorithm based on various factors such as integral representations, exponential asymptotics among various others to evaluate m-parameter Mittag-Leffler function, study its behaviour as holomorphic function and its dependence upon the parameters $\alpha, \rho, \beta_j, \kappa_j, \mu_i$

and ν_i for $i = 1, ..., r$; $j = 1, ..., p$.

3 Generalized Fractional Kinetic Equation

Here, we investigate the solutions of generalized fractional kinetic equations involving m-parameter Mittag-Leffler function. The solutions are obtained in terms of Wiman function and m-parameter Mittag-Leffler function using Natural transform.

Theorem 1. *Let $(\boldsymbol{\mu}, \boldsymbol{\nu})_r = [\mu_1, \nu_1; \mu_2, \nu_2; \ldots; \mu_r, \nu_r]$, $(\boldsymbol{\beta}, \boldsymbol{\kappa})_p = [\beta_1, \kappa_1; \beta_2, \kappa_2; \ldots; \beta_p, \kappa_p]$, $r+p = m-2$, m is any positive integer. If $\mu_i, \nu_i, \alpha, \rho, \beta_j, \kappa_j \in \mathbb{C}$, with $\min \Re\{\alpha, \rho, \mu_i, \nu_i, \beta_j, \kappa_j\} > 0$ for $i = 1, ..., r$; $j = 1, ..., p$, $q > 0$ and $\xi > 0$, then the solution of the equation*

$$\mathfrak{M}(\tau) - \mathfrak{M}_0 E^{(\boldsymbol{\mu},\boldsymbol{\nu})_r}_{\alpha,\rho;(\boldsymbol{\beta},\boldsymbol{\kappa})_p}(\tau) = -q^{\xi} {}_0\mathfrak{D}_{\tau}^{-\xi} \mathfrak{M}(\tau) \tag{20}$$

is given by

$$\mathfrak{M}(\tau) = \mathfrak{M}_0 E^{(\boldsymbol{\mu},\boldsymbol{\nu})_r}_{\alpha,\rho;(\boldsymbol{\beta},\boldsymbol{\kappa})_p}(\tau) E_{\xi,n+1}(-q^{\xi}\tau^{\xi})\Gamma(n+1). \tag{21}$$

Proof. Re-arranging the equation (20), we have

$$\mathfrak{M}(\tau) = \mathfrak{M}_0 \sum_{n=0}^{\infty} \frac{(\mu_1)_{\nu_1 n}(\mu_2)_{\nu_2 n}\ldots(\mu_r)_{\nu_r n}}{\Gamma(\alpha n+\rho)(\beta_1)_{\kappa_1 n}\ldots(\beta_p)_{\kappa_p n}} \tau^n - q^{\xi} {}_0\mathfrak{D}_{\tau}^{-\xi}\mathfrak{M}(\tau). \tag{22}$$

The Natural transform of Riemann-Liouville integral operator is:

$$N[{}_0\mathfrak{D}_{\tau}^{-\xi} g(\tau); s, u] = u^{\xi} s^{-\xi} G(s,u), \tag{23}$$

where $G(s,u)$ is the Natural transform of $g(\tau)$.
Applying Natural transforms on both sides of equation (22), we get

$$\begin{aligned} &N[\mathfrak{M}(\tau); s, u] \\ &= \mathfrak{M}_0 \left(\int_0^{\infty} e^{-s\tau} \sum_{n=0}^{\infty} \frac{(\mu_1)_{\nu_1 n}(\mu_2)_{\nu_2 n}\ldots(\mu_r)_{\nu_r n}}{\Gamma(\alpha n+\rho)(\beta_1)_{\kappa_1 n}\ldots(\beta_p)_{\kappa_p n}} u^{n+1}\tau^n d\tau \right) - q^{\xi} N\left[{}_0\mathfrak{D}_{\tau}^{-\xi}\mathfrak{M}(\tau); s, u\right] \end{aligned}$$

$$\implies \mathfrak{M}(s,u) = \mathfrak{M}_0 \left(\sum_{n=0}^{\infty} \frac{(\mu_1)_{\nu_1 n}(\mu_2)_{\nu_2 n}\ldots(\mu_r)_{\nu_r n}}{\Gamma(\alpha n+\rho)(\beta_1)_{\kappa_1 n}\ldots(\beta_p)_{\kappa_p n}} \int_0^{\infty} e^{-s\tau} u^{n+1}\tau^n d\tau \right) - q^{\xi} u^{\xi} s^{-\xi}\mathfrak{M}(s,u)$$

$$\implies \mathfrak{M}(s,u)\left[1 + q^{\xi}u^{\xi}s^{-\xi}\right] = \mathfrak{M}_0 \sum_{n=0}^{\infty} \frac{(\mu_1)_{\nu_1 n}(\mu_2)_{\nu_2 n}\ldots(\mu_r)_{\nu_r n}}{\Gamma(\alpha n+\rho)(\beta_1)_{\kappa_1 n}\ldots(\beta_p)_{\kappa_p n}} \frac{\Gamma(n+1)}{s^{n+1}} u^{n+1}$$

$$\implies \mathfrak{M}(s,u) = \mathfrak{M}_0 \sum_{n=0}^{\infty} \frac{(\mu_1)_{\nu_1 n}(\mu_2)_{\nu_2 n}\ldots(\mu_r)_{\nu_r n}}{\Gamma(\alpha n+\rho)(\beta_1)_{\kappa_1 n}\ldots(\beta_p)_{\kappa_p n}} \Gamma(n+1) s^{-(n+1)} u^{n+1} \sum_{l=0}^{\infty} \left[- \left(\frac{s}{qu}\right)^{-\xi} \right]^l . \tag{24}$$

Taking the inverse Natural transform of the above equation (24) and using $N^{-1}\left[\left(\frac{s}{u}\right)^{-\xi}; \tau\right] =$

$\frac{\tau^{\xi-1}}{\Gamma(\xi)}$, $\Re(\xi)>0$, we get

$$\begin{aligned}
&N^{-1}\left[\mathfrak{M}(s,u)\right]\\
&=\mathfrak{M}_0\sum_{n=0}^{\infty}\frac{(\mu_1)_{\nu_1 n}(\mu_2)_{\nu_2 n}\ldots(\mu_r)_{\nu_r n}}{\Gamma(\alpha n+\rho)(\beta_1)_{\kappa_1 n}\ldots(\beta_p)_{\kappa_p n}}\Gamma(n+1)N^{-1}\left[\sum_{l=0}^{\infty}(-1)^l q^{\xi l}u^{\xi l+n+1}s^{-(n+\xi l+1)}\right]\\
&\implies \mathfrak{M}(\tau)=\mathfrak{M}_0\sum_{n=0}^{\infty}\frac{(\mu_1)_{\nu_1 n}(\mu_2)_{\nu_2 n}\ldots(\mu_r)_{\nu_r n}}{\Gamma(\alpha n+\rho)(\beta_1)_{\kappa_1 n}\ldots(\beta_p)_{\kappa_p n}}\Gamma(n+1)\sum_{l=0}^{\infty}(-1)^l q^{\xi l}\frac{\tau^{n+\xi l}}{\Gamma(n+\xi l+1)}\\
&\implies \mathfrak{M}(\tau)=\mathfrak{M}_0\sum_{n=0}^{\infty}\frac{(\mu_1)_{\nu_1 n}(\mu_2)_{\nu_2 n}\ldots(\mu_r)_{\nu_r n}}{\Gamma(\alpha n+\rho)(\beta_1)_{\kappa_1 n}\ldots(\beta_p)_{\kappa_p n}}\tau^n\Gamma(n+1)\sum_{l=0}^{\infty}(-1)^l\frac{(q\tau)^{\xi l}}{\Gamma(n+\xi l+1)}\\
&\implies \mathfrak{M}(\tau)=\mathfrak{M}_0E^{(\boldsymbol{\mu},\boldsymbol{\nu})_r}_{\alpha,\rho;(\boldsymbol{\beta},\boldsymbol{\kappa})_p}(\tau)E_{\xi,n+1}(-q^\xi\tau^\xi)\Gamma(n+1).
\end{aligned}$$

Hence we get our desired result (21).

□

Theorem 2. *If $\mu_i,\nu_i,\alpha,\rho,\beta_j,\kappa_j\in\mathbb{C}$, with $\min\Re\{\alpha,\rho,\mu_i,\nu_i,\beta_j,\kappa_j\}>0$ for $i=1,...,r$; $j=1,...,p$, $q>0$ and $\xi>0$, then the solution of the equation*

$$\mathfrak{M}(\tau)-\mathfrak{M}_0E^{(\boldsymbol{\mu},\boldsymbol{\nu})_r}_{\alpha,\rho;(\boldsymbol{\beta},\boldsymbol{\kappa})_p}(q^\xi\tau^\xi)=-q^\xi{}_0\mathfrak{D}^{-\xi}_\tau\mathfrak{M}(\tau) \tag{25}$$

is given by

$$\mathfrak{M}(\tau)=\mathfrak{M}_0E^{(\boldsymbol{\mu},\boldsymbol{\nu})_r}_{\alpha,\rho;(\boldsymbol{\beta},\boldsymbol{\kappa})_p}(q^\xi\tau^\xi)E_{\xi,\xi n+1}(-q^\xi\tau^\xi)\Gamma(\xi n+1). \tag{26}$$

Proof. Re-arranging the equation (25), we have

$$\mathfrak{M}(\tau)=\mathfrak{M}_0\sum_{n=0}^{\infty}\frac{(\mu_1)_{\nu_1 n}(\mu_2)_{\nu_2 n}\ldots(\mu_r)_{\nu_r n}}{\Gamma(\alpha n+\rho)(\beta_1)_{\kappa_1 n}\ldots(\beta_p)_{\kappa_p n}}q^{\xi n}\tau^{\xi n}-q^\xi{}_0\mathfrak{D}^{-\xi}_\tau\mathfrak{M}(\tau). \tag{27}$$

Applying Natural transforms on both sides of equation (27), we get

$$\begin{aligned}
&N[\mathfrak{M}(\tau);s,u]\\
&=\mathfrak{M}_0\left(\int_0^\infty e^{-s\tau}\sum_{n=0}^{\infty}\frac{(\mu_1)_{\nu_1 n}(\mu_2)_{\nu_2 n}\ldots(\mu_r)_{\nu_r n}}{\Gamma(\alpha n+\rho)(\beta_1)_{\kappa_1 n}\ldots(\beta_p)_{\kappa_p n}}q^{\xi n}u^{\xi n+1}\tau^{\xi n}d\tau\right)-q^\xi N\left[{}_0\mathfrak{D}^{-\xi}_\tau\mathfrak{M}(\tau);s,u\right]\\
&\implies \mathfrak{M}(s,u)\\
&=\mathfrak{M}_0\left(\sum_{n=0}^{\infty}\frac{(\mu_1)_{\nu_1 n}(\mu_2)_{\nu_2 n}\ldots(\mu_r)_{\nu_r n}}{\Gamma(\alpha n+\rho)(\beta_1)_{\kappa_1 n}\ldots(\beta_p)_{\kappa_p n}}q^{\xi n}u^{\xi n+1}\int_0^\infty e^{-s\tau}\tau^{\xi n}d\tau\right)-q^\xi u^\xi s^{-\xi}\mathfrak{M}(s,u)\\
&\implies \mathfrak{M}(s,u)\left[1+q^\xi u^\xi s^{-\xi}\right]=\mathfrak{M}_0\sum_{n=0}^{\infty}\frac{(\mu_1)_{\nu_1 n}(\mu_2)_{\nu_2 n}\ldots(\mu_r)_{\nu_r n}}{\Gamma(\alpha n+\rho)(\beta_1)_{\kappa_1 n}\ldots(\beta_p)_{\kappa_p n}}q^{\xi n}\frac{\Gamma(\xi n+1)}{s^{\xi n+1}}u^{\xi n+1}
\end{aligned}$$

$$\implies \mathfrak{M}(s,u)=\mathfrak{M}_0\sum_{n=0}^{\infty}\frac{(\mu_1)_{\nu_1 n}(\mu_2)_{\nu_2 n}\ldots(\mu_r)_{\nu_r n}}{\Gamma(\alpha n+\rho)(\beta_1)_{\kappa_1 n}\ldots(\beta_p)_{\kappa_p n}}\Gamma(\xi n+1)q^{\xi n}u^{\xi n+1}s^{-(\xi n+1)}\sum_{l=0}^{\infty}\left[-\left(\frac{s}{qu}\right)^{-\xi}\right]^l. \tag{28}$$

Taking the inverse Natural transform of the above equation (28). Using $N^{-1}\left[\left(\frac{s}{u}\right)^{-\xi};\tau\right]=\frac{\tau^{\xi-1}}{\Gamma(\xi)}$, $\Re(\xi)>0$ and simplifying as in Theorem 1 we get

$$\mathfrak{M}(\tau)=\mathfrak{M}_0E^{(\boldsymbol{\mu},\boldsymbol{\nu})_r}_{\alpha,\rho;(\boldsymbol{\beta},\boldsymbol{\kappa})_p}(q^\xi\tau^\xi)E_{\xi,\xi n+1}(-q^\xi\tau^\xi)\Gamma(\xi n+1).$$

Hence we get our desired result (26).

□

Theorem 3. *If $\mu_i, \nu_i, \alpha, \rho, \beta_j, \kappa_j \in \mathbb{C}$, with $\min \Re\{\alpha, \rho, \mu_i, \nu_i, \beta_j, \kappa_j\} > 0$ for $i = 1, ..., r$; $j = 1, ..., p$, $q > 0$ and $\xi > 0$ and $\varphi > 0$, then the solution of the equation*

$$\mathfrak{M}(\tau) - \mathfrak{M}_0 E^{(\boldsymbol{\mu},\boldsymbol{\nu})_r}_{\alpha,\rho;(\boldsymbol{\beta},\boldsymbol{\kappa})_p}(q^\xi \tau^\xi) = -\varphi^\xi {}_0\mathfrak{D}_\tau^{-\xi} \mathfrak{M}(\tau) \tag{29}$$

is given by

$$\mathfrak{M}(\tau) = \mathfrak{M}_0 E^{(\boldsymbol{\mu},\boldsymbol{\nu})_r}_{\alpha,\rho;(\boldsymbol{\beta},\boldsymbol{\kappa})_p}(q^\xi \tau^\xi) E_{\xi,\xi n+1}(-\varphi^\xi \tau^\xi) \Gamma(\xi n + 1). \tag{30}$$

Proof. Re-arranging the equation (29), we have

$$\mathfrak{M}(\tau) = \mathfrak{M}_0 \sum_{n=0}^{\infty} \frac{(\mu_1)_{\nu_1 n}(\mu_2)_{\nu_2 n} \dots (\mu_r)_{\nu_r n}}{\Gamma(\alpha n + \rho)(\beta_1)_{\kappa_1 n} \dots (\beta_p)_{\kappa_p n}} q^{\xi n} \tau^{\xi n} - \varphi^\xi {}_0\mathfrak{D}_\tau^{-\xi} \mathfrak{M}(\tau). \tag{31}$$

Applying Natural transforms on both sides of equation (31), we get

$$\begin{aligned} &N[\mathfrak{M}(\tau); s, u] \\ &= \mathfrak{M}_0 \left(\int_0^\infty e^{-s\tau} \sum_{n=0}^{\infty} \frac{(\mu_1)_{\nu_1 n}(\mu_2)_{\nu_2 n} \dots (\mu_r)_{\nu_r n}}{\Gamma(\alpha n + \rho)(\beta_1)_{\kappa_1 n} \dots (\beta_p)_{\kappa_p n}} q^{\xi n} u^{\xi n+1} \tau^{\xi n} d\tau \right) - \varphi^\xi N \left[{}_0\mathfrak{D}_\tau^{-\xi} \mathfrak{M}(\tau); s, u \right] \end{aligned}$$

$$\begin{aligned} \implies &\mathfrak{M}(s, u) \\ &= \mathfrak{M}_0 \left(\sum_{n=0}^{\infty} \frac{(\mu_1)_{\nu_1 n}(\mu_2)_{\nu_2 n} \dots (\mu_r)_{\nu_r n}}{\Gamma(\alpha n + \rho)(\beta_1)_{\kappa_1 n} \dots (\beta_p)_{\kappa_p n}} q^{\xi n} u^{\xi n+1} \int_0^\infty e^{-s\tau} \tau^{\xi n} d\tau \right) - \varphi^\xi u^\xi s^{-\xi} \mathfrak{M}(s, u) \end{aligned}$$

$$\implies \mathfrak{M}(s, u) \left[1 + \varphi^\xi u^\xi s^{-\xi} \right] = \mathfrak{M}_0 \sum_{n=0}^{\infty} \frac{(\mu_1)_{\nu_1 n}(\mu_2)_{\nu_2 n} \dots (\mu_r)_{\nu_r n}}{\Gamma(\alpha n + \rho)(\beta_1)_{\kappa_1 n} \dots (\beta_p)_{\kappa_p n}} q^{\xi n} \frac{\Gamma(\xi n + 1)}{s^{\xi n+1}} u^{\xi n+1}$$

$$\implies \mathfrak{M}(s, u) = \mathfrak{M}_0 \sum_{n=0}^{\infty} \frac{(\mu_1)_{\nu_1 n}(\mu_2)_{\nu_2 n} \dots (\mu_r)_{\nu_r n}}{\Gamma(\alpha n + \rho)(\beta_1)_{\kappa_1 n} \dots (\beta_p)_{\kappa_p n}} \Gamma(\xi n + 1) q^{\xi n} u^{\xi n+1} s^{-(\xi n+1)} \sum_{l=0}^{\infty} \left[- \left(\frac{s}{\varphi u} \right)^{-\xi} \right]^l. \tag{32}$$

Taking the inverse Natural transform of the above equation (32). Using $N^{-1}\left[\left(\frac{s}{u}\right)^{-\xi}; \tau\right] = \frac{\tau^{\xi-1}}{\Gamma(\xi)}$, $\Re(\xi) > 0$ and simplifying as in Theorem 1 we get

$$\mathfrak{M}(\tau) = \mathfrak{M}_0 E^{(\boldsymbol{\mu},\boldsymbol{\nu})_r}_{\alpha,\rho;(\boldsymbol{\beta},\boldsymbol{\kappa})_p}(q^\xi \tau^\xi) E_{\xi,\xi n+1}(-\varphi^\xi \tau^\xi) \Gamma(\xi n + 1).$$

Hence we get our desired result (30).

□

3.1 Special cases.

If $r = p = 0$ and $\rho = 1$, Theorem 1 gives the fractional kinetic equation for Gosta Mittag-Leffler function (11):

Corollary 1. *If $\alpha \geq 0$, $q > 0$ and $\xi > 0$, then the solution of the equation*

$$\mathfrak{M}(\tau) - \mathfrak{M}_0 E_\alpha(\tau) = -q^\xi {}_0\mathfrak{D}_\tau^{-\xi} \mathfrak{M}(\tau) \tag{33}$$

is given by

$$\mathfrak{M}(\tau) = \mathfrak{M}_0 E_\alpha(\tau) E_{\xi,n+1}(-q^\xi \tau^\xi) \Gamma(n + 1). \tag{34}$$

If $r = p = 0$, Theorem 1 gives the fractional kinetic equation for Wiman function (12) :

Corollary 2. *If $\alpha, \rho \in \mathbb{C}, \Re(\alpha) > 0, \Re(\rho) > 0$, $q > 0$ and $\xi > 0$, then the solution of the equation*

$$\mathfrak{M}(\tau) - \mathfrak{M}_0 E_{\alpha,\rho}(\tau) = -q^{\xi}{}_0\mathfrak{D}_{\tau}^{-\xi}\mathfrak{M}(\tau) \tag{35}$$

is given by

$$\mathfrak{M}(\tau) = \mathfrak{M}_0 E_{\alpha,\rho}(\tau) E_{\xi,n+1}(-q^{\xi}\tau^{\xi})\Gamma(n+1). \tag{36}$$

If $r = p = 1$ and $\nu_1 = 1, \beta_1 = 1, \kappa_1 = 1$, Theorem 1 gives the fractional kinetic equation for Prabhakar function (13) :

Corollary 3. *If $\Re(\alpha) > 0, \Re(\rho) > 0, \mu > 0$, $q > 0$ and $\xi > 0$, then the solution of the equation*

$$\mathfrak{M}(\tau) - \mathfrak{M}_0 E^{\mu}_{\alpha,\rho}(\tau) = -q^{\xi}{}_0\mathfrak{D}_{\tau}^{-\xi}\mathfrak{M}(\tau) \tag{37}$$

is given by

$$\mathfrak{M}(\tau) = \mathfrak{M}_0 E^{\mu}_{\alpha,\rho}(\tau) E_{\xi,n+1}(-q^{\xi}\tau^{\xi})\Gamma(n+1). \tag{38}$$

If $r = p = 1$ and $\beta_1 = 1, \kappa_1 = 1$, Theorem 1 gives the fractional kinetic equation for the generalization of Prabhakar function given by Shukla and Prajapati (14):

Corollary 4. *If $\min\{\Re(\alpha), \Re(\rho), \Re(\mu)\} > 0$, $\alpha, \rho, \mu, \nu \in \mathbb{C}$, $q > 0$ and $\xi > 0$, then the solution of the equation*

$$\mathfrak{M}(\tau) - \mathfrak{M}_0 E^{\mu,\nu}_{\alpha,\rho}(\tau) = -q^{\xi}{}_0\mathfrak{D}_{\tau}^{-\xi}\mathfrak{M}(\tau) \tag{39}$$

is given by

$$\mathfrak{M}(\tau) = \mathfrak{M}_0 E^{\mu,\nu}_{\alpha,\rho}(\tau) E_{\xi,n+1}(-q^{\xi}\tau^{\xi})\Gamma(n+1). \tag{40}$$

If $r = p = 1$ and $\kappa_1 = 1$, Theorem 1 gives the fractional kinetic equation for the generalization given by Khan and Ahmed (15) :

Corollary 5. *If $\min\{\Re(\alpha), \Re(\rho), \Re(\beta), \Re(\mu)\} > 0$, $\nu \in (0,1) \cup \mathbb{N}$, $q > 0$ and $\xi > 0$, then the solution of the equation*

$$\mathfrak{M}(\tau) - \mathfrak{M}_0 E^{\mu,\nu}_{\alpha,\rho,\beta}(\tau) = -q^{\xi}{}_0\mathfrak{D}_{\tau}^{-\xi}\mathfrak{M}(\tau) \tag{41}$$

is given by

$$\mathfrak{M}(\tau) = \mathfrak{M}_0 E^{\mu,\nu}_{\alpha,\rho,\beta}(\tau) E_{\xi,n+1}(-q^{\xi}\tau^{\xi})\Gamma(n+1). \tag{42}$$

If $r = p = 2$, Theorem 1 gives the fractional kinetic equation for the further generalization given by Khan and Ahmed (16):

Corollary 6. *If $\alpha, \rho, \mu_1, \nu_1, \mu_2, \beta_1, \kappa_1, \beta_2 \in \mathbb{C}$, $\nu_2, \kappa_2 > 0$, $\Re(\alpha) + \kappa_2 \geq \nu_2$ $\min\{\Re(\alpha), \Re(\rho), \Re(\mu_1), \Re(\nu_1), \Re(\mu_2), \Re(\beta_1), \Re(\kappa_1), \Re(\beta_2)\} > 0$, $q > 0$ and $\xi > 0$, then the solution of the equation*

$$\mathfrak{M}(\tau) - \mathfrak{M}_0 E^{\mu_1,\nu_1,\mu_2,\nu_2}_{\alpha,\rho,\beta_1,\kappa_1,\beta_2,\kappa_2}(\tau) = -q^{\xi}{}_0\mathfrak{D}_{\tau}^{-\xi}\mathfrak{M}(\tau) \tag{43}$$

is given by

$$\mathfrak{M}(\tau) = \mathfrak{M}_0 E^{\mu_1,\nu_1,\mu_2,\nu_2}_{\alpha,\rho,\beta_1,\kappa_1,\beta_2,\kappa_2}(\tau) E_{\xi,n+1}(-q^{\xi}\tau^{\xi})\Gamma(n+1). \tag{44}$$

Remark 1. *By taking suitable conditions and particular values of the parameters in Theorems 2 and 3 , we can reduce the generalized fractional kinetic equations involving m-parameter Mittag-Leffler function to fractional kinetic equations involving various types of Mittag-Leffler functions in a similar way as the above corollaries.*

4 Conclusion

In this article we have simulated the complex and contour plots of some particular cases of m-parameter Mittag-Leffler function using MATHEMATICA-12. Several other plots can be obtained and discussed on changing the values of the parameters. This will help in developing a numerical algorithm which will be used to evaluate the m-parameter Mittag-Leffler function. The algorithm will also help in studying its behaviour as holomorphic function and its dependence upon the parameters $\alpha, \rho, \beta_j, \kappa_j, \mu_i$ and ν_i for $i = 1, ..., r$; $j = 1, ..., p$. We have also introduced a new fractional generalization of the standard kinetic equation involving m-parameter Mittag-Leffler function and derived the solutions for the same. Under some special conditions, we have reduced the generalized fractional kinetic equation involving m-parameter Mittag-Leffler function to fractional kinetic equation involving Mittag-Leffler, Wiman, Prabhakar function and various other forms of Mittag-Leffler function. The results that we have obtained is general in nature. We can find several new and known solutions of fractional kinetic equations involving some other function.

5 Acknowledgement

The authors would like to thank the reviewers for their valuable suggestions.

References

Agarwal, P., Chand, M., Baleanu, D., & Jain, S. (2018). On the solutions of certain fractional kinetic equations involving k-Mittag-Leffler function. *Adv. Difference Equ.*, *2018*, 249.

Agarwal, R., Chandola, A., Pandey, R.M., & Nisar, K.S. (2021). m-parameter Mittag–Leffler function, its various properties, and relation with fractional calculus operators. *Math. Methods Appl. Sci.*, *44*(7), 5365–5384. https://doi.org/10.1002/mma.7115.

Atangana, A. (2017). Fractal-fractional differentiation and integration: connecting fractal calculus and fractional calculus to predict complex system. *Chaos Solitons Fractals*, *102*, 396–406.

Chandola, A., Pandey, R.M., Agarwal, R., & Purohit, S.D. (2020). An extension of beta function, its statistical distribution and associated fractional operator. *Adv. Difference Equ.*, *2020*, 684. https://doi.org/10.1186/s13662-020-03142-6.

Chandola, A., Pandey R.M., & Agarwal, R. (2021a). Bessel-maitland function of several variables and its properties related to integral transforms and fractional calculus. *Applications and Applied Mathematics: An International Journal*, *16*(1), Article 23. https://digitalcommons.pvamu.edu/aam/vol16/iss1/23.

Chandola, A., Pandey R.M., & Agarwal, R. (2021b). A study of integral transform of convolution type integrals involving k-hypergeometric functions. *Math. Eng. Sci. Aerosp.*, *12*(3), 795–810.

Ghanbari, B., Kumar, S., and Kumar, R. (2020). A study of behaviour for immune and tumor cells in immunogenetic tumour model with non-singular fractional derivative. *Chaos Solitons Fractals*, *133*. https://doi.org/10.1016/j.chaos.2020.109619.

Haubold, H., Mathai, A. (2000). The fractional kinetic equation and thermonuclear functions. *Astrophys. Space Sci.*, *327*, 53–63.

Khan, M., Ahmed, S. (2013). On some properties of the generalized Mittag-Leffler function. *Springerplus*, *2*, 337.

Khan, Z., Khan, W. (2008). N-transform-properties and applications. *NUST J. Eng. Sci.*, *1*(1), 127–133.

Kilbas, A.A., M.S.H., & Trujillo, J.J. (2006). *Theory and Applications of Fractional Differential Equations*. Elsevier, Amsterdam.

Kumar, S., Nisar, K.S., Kumar, R., Cattani, C., & Samet, B. (2020). A new Rabotnov fractional-exponential function based fractional derivative for diffusion equation under external force. *Math. Methods Appl. Sci.*, 1–12. https://doi.org/10.1002/mma.6208.

Liouville, J. (1823). Memoire: sur le calcul des differentielles a indices quelconquessur. *J. de l'École polytechnique*, *13*(21), 71–162.

Mittag-Leffler, G. (1903). Sur la nouvelle fonction $E_\alpha(x)$. *C. R. Acad. Sci. Paris, (Ser. II)*, *137*, 554–558.

Nisar, K., Suthar, D., Agarwal, R., & Purohit, S. (2020). Fractional calculus operators with appell function kernels applied to Srivastava polynomials and extended Mittag-Leffler function. *Adv. Difference Equ.*, *2020*, 148.

Nisar, K.S. (2020). Generalized Mittag-Leffler type function: Fractional integrations and application to fractional kinetic equations. *Front. Phys.*, *8*33.

Prabhakar, T.R. (1971). A singular integral equation with a generalized Mittag-Leffler function in the kernel. *Yokohama Math. J.*, *19*, 7–15.

Rainville, E. D. (1971). *Special Functions*. Chelsea Publications, Bronx, New York.

Saxena, R.K., Kalla, S.L. (2008). On the solutions of certain fractional kinetic equations. *Appl. Math. Comput.*, *199*, 504–511.

Shukla, A.K., Prajapati, J.C. (2007). On a generalization of Mittag-Leffler function and its properties. *J. Math. Anal. Appl.*, *336*, 797–811.

Spiegel, M. (1965). *Theory and Problems of Laplace Transforms*. Schaums Outline Series. McGraw-Hill, New York.

Watugala, G. (1993). Sumudu transform: a new integral transform to solve differential equations and control engineering problems. *Internat. J. Math. Ed. Sci. Tech.*, *24*(1), 35–43.

Wiman, A. (1905). Über den fundamentalsatz in der teorie der funktionen $E_a(x)$. *Acta Math.*, *29*, 191–201.